A PHENOMENOLOGICAL MODEL OF PASSIVE CONTROL OF VIBRATIONS OF BEAMS VIA SHUNTED PIEZOELECTRIC TRANSDUCERS

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ABSTRACT

A phenomenological electromechanical analytical model of beams with piezoelectric transducers shunted with a passive electrical network with general impedance is presented and discussed. A case study is considered and the damping performance of a cantilever beam with a piezoelectric patch shunted with a resistance tuned to damp the first mode is investigated.

1. INTRODUCTION

The use of piezoelectric materials for the active vibration control (AVC) of flexible structures has been widely studied in the last two decades [Clark *et al.* (1998), Preumont (2002)]. Besides allowing a direct connection with an input/output electrical system, these materials are well adapted to distributed control of vibrations, since they are produced as very thin patches that can be bonded or embedded in structures.

The idea of connecting piezoelectric patches to shunt circuits to control the mechanical vibration takes advantage of the direct piezoelectric effect and consists, basically, in dissipating the mechanically induced electrical energy in the piezoelectric patch by coupling it together with an electrical network with resistive, capacitive and inductive capabilities [Hagood and von Flotow (1991), Moheimani and Fleming (2006)].

In this article a coupled electromechanical model of a beam submitted to a harmonic force with a piezoelectric patch bonded on its surface is present and the shunted damping performance is investigated.

2. MECHANICAL MODEL

Following the well known Euler-Bernoulli assumptions for thin beams, the transverse vibration of beams with a constant cross sectional area is governed by [Timoshenko *et al.* (1974)]

$$E_b I_b \frac{\partial^4 w(x,t)}{\partial x^4} + \rho_b A_b \frac{\partial^2 w(x,t)}{\partial t^2} = p(x,t) , \qquad (1)$$

where x and t are the spatial and time coordinates, $w(x,t)$ is the transverse displacement, E_b is the Young's modulus, A_b is the cross sectional area, I_b is the second-order moment of area and ρ_b the mass density of the beam.

The solution of the governing equation of motion in (1) might be obtained by performing a modal projection where the response is assumed to be expressed as a linear combination of the natural mode shapes,

$$w(x,t) = \sum_{r=1}^{n} \varphi_r(x)\eta_r(t)\,, \tag{2}$$

where $\varphi_r(x)$ and $\eta_r(t)$ are the rth mode shape and modal coordinate, respectively, and n the number of modes considered in the analysis.

3. SENSING AND ACTUATION

For piezoelectric materials of the crystal class mm2 polarized in the transverse direction, the one-dimensional actuating and sensing constitutive behaviors are given by

$$\sigma_{11} = E_p\varepsilon_{11} + e_{31}E_3,\quad D_3 = e_{31}\varepsilon_{11} - \epsilon_{33}^{T}\, E_3, \tag{3}$$

where E_p is the Young's modulus under constant electric field, e_{31} is the piezoelectric stress constant, ϵ_{33}^{T} is the dielectric constant under constant stress and D_3 and E_3 are the electric displacement and electric field in the transverse direction, respectively [Preumont (2006)].

According to the Euler-Bernoulli assumptions, the extensional strain component is given by

$$\varepsilon_{11}(x,z,t) = -z\frac{\partial^2 w(x,t)}{\partial x^2}, \tag{4}$$

where z is the transverse spatial coordinate which starts at the beam neutral axis.

Neglecting the higher-order components of the electric potential the electric field can be given as

$$E_3(t) = -\frac{v(t)}{2h_p}\,, \tag{5}$$

where $v(t)$ is the electric potential difference between the two electrodes and h_p the piezoelectric patch thickness. Furthermore, the effects of the piezoelectric actuation are equivalent to considering two concentrated moments $m_p(t)$ with opposite senses and value equal to

$$m_p(t) = -\int_{A_p} e_{31}\frac{v(t)}{2h_p}\mathrm{d}A_p = -C_m v(t), \tag{6}$$

where

$C_m = e_{31}b_p(h_b + h_p)$,

A_p is the cross sectional area, b_p is the width and $2h_p$ the thickness of the of the piezoelectric transducer, and $2h_b$ the beam's thickness.

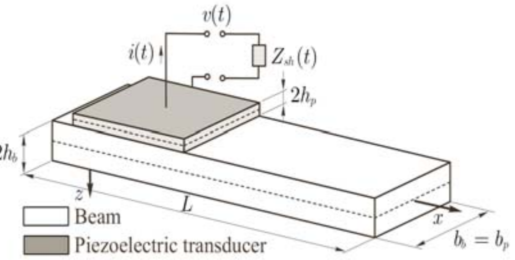

Fig 1: Generic beam structure with a shunted piezoelectric transducer.

Conversely, the electrical displacement is related with the enforced strain as defined in the second of Eqs. (3). Assuming, for simplicity, that the induced extensional strain in the piezoelectric is constant and equal to $\varepsilon_{11}(x, z = h_b, t)$, integrating the electrical displacement over the electrode area A_e and considering the mechanical strain definition in Eq. (4), the induced electric charge $q(t)$ is given by

$$\begin{aligned} q(t) &= -\int_{A_e}\left[e_{31}z\frac{\partial^2 w(x,t)}{\partial x^2} + \epsilon_{33}^{T}\frac{v(t)}{2h_p}\right]\mathrm{d}A_e \\ &= -e_{31}h_b b_p\left[\frac{\partial w(x,t)}{\partial x}\right]\Bigg|_a^b - C_p v(t), \end{aligned} \tag{7}$$

where the piezoelectric patch is comprised between $x = a$ and $x = b$, with $a < b$ and length L_p, and the piezoelectric capacitance is given by

$$C_p = \frac{\epsilon_{33}^{T}\, A_e}{2h_p}\,. \tag{8}$$

By last, the current $i(t) = \mathrm{d}q(t)/\mathrm{d}t$ is given by

$$i(t) = -C_q\left[\frac{\partial^2 w(x,t)}{\partial x\partial t}\right]\Bigg|_a^b - C_p\frac{\mathrm{d}v(t)}{\mathrm{d}t}\,, \tag{9}$$

where $C_q = e_{31}h_b b_p$.

4. PIEZOELECTRIC SHUNTED DAMPING

As can be seen in Eq. (9), the piezoelectric transducer can be equivalently represented by an electrical network comprising a capacitor in series with a voltage generator which is dynamically dependent on the induced mechanical strains, which are coupled together due to the direct piezoelectric effect.

The Laplace transform of Eq. (9) is

$$i(s) = -sC_q \left[\frac{\partial w(x,s)}{\partial x} \right]\Bigg|_a^b - \frac{1}{Z_{oc}(s)} v(s)\,, \qquad (10)$$

where $Z_{oc}(s) = (sC_p)^{-1}$ is the open-circuit electrical impedance and s is the Laplace's complex variable.

When an electrical network with impedance $Z_{sh}(s)$ is added in parallel (shunted) to the piezoelectric patch, the total electrical impedance $Z_{el}(s)$ is determined from the relationship

$$\frac{1}{Z_{el}(s)} = \frac{1}{Z_{oc}(s)} + \frac{1}{Z_{sh}(s)}\,, \qquad (11)$$

where

$$Z_{el}(s) = \frac{Z_{oc}(s)Z_{sh}(s)}{Z_{oc}(s)+Z_{sh}(s)}\,. \qquad (12)$$

Using this expression in Eq. (10), the voltage appearing across the electrodes is given by

$$v(s) = -Z_{el}(s)\left\{ -i(s) - sC_q \left[\frac{\partial w(x,s)}{\partial x} \right]\Bigg|_a^b \right\}. \qquad (13)$$

When substituted back into Eq. (6) and considering the prescribed current $i(t)=0$, the moment $m_p(s)$ is given by

$$m_p(s) = -C_m C_q s Z_{el}(s) \left[\frac{\partial w(x,s)}{\partial x} \right]\Bigg|_a^b . \qquad (14)$$

Eq. (14) represents the equivalent moment effects due to the piezoelectric shunted electrical network. Depending of the passive network design and electrical boundary conditions (EBCs), different damping performances can be achieved, which can be interpreted in what follows:

- Closed-circuit (short-circuit) EBC:

$$Z_{sh} = 0,\;\; Z_{el} = 0,\;\; m_p(s) = 0\,; \qquad (15)$$

- Open-circuit EBC:

$$Z_{sh} \to \infty,\;\; Z_{el} = Z_{oc},\;\; m_p(s) \neq 0\,. \qquad (16)$$

5. ELECTROMECHANICAL MODEL

The beam's mechanical model in Eq. (1) has considered a generic loading term $p(x,t)$. In general, that term might consider the loading effects of both distributed forces $f(x,t)$ and/or moments $m(x,t)$, so that [Preumont (2006)]

$$p(x,t) = f(x,t) + \frac{\partial^2 m(x,t)}{\partial x^2}\,. \qquad (17)$$

Let us consider a concentrated disturbance force,

$$f(x,t) = \delta(x - x_F)F(t)\,, \qquad (18)$$

where $\delta(x)$ is the Dirac delta function. Since the actuating effects of a piezoelectric patch bonded in a beam might be equivalently considered through two concentrated moments with opposite signs at its boundaries [Preumont (2006)], the seond term in the right-hand side of Eq. (17) can be used to introduce the actuating effects of a piezoelectric patch, which might be defined as

$$m(x,t) = m_p(t)\left[H(x-a) - H(x-b)\right], \qquad (19)$$

where $H(x)$ is the Heaviside function and $m_p(t)$ is the prescribed distributed moment as defined in Eq. (14). Considering Eq.(19), the Laplace transform of the second spatial derivative of $m(x,t)$ is

$$\frac{\partial^2 m(x,s)}{\partial x^2} = m_p(s)[\delta'(x-a) - \delta'(x-b)]\,, \qquad (20)$$

where $\delta'(x)$ is the spatial derivative of the Dirac delta function.

Considering the modal projection defined in Eq. (2) and substituting into Eq. (1), taking into consideration the orthonormality properties of the mode shapes, the equation of motion in the Laplace domain for the rth mode can be written as

$$(s^2 + 2\xi\omega_r s + \omega_r{}^2)\eta_r(s) = N_r(x,s)\,, \qquad (21)$$

where the loading term $N_r(x,s)$ is given by

$$N_r(x,s) = \int_L \varphi_r(x)\,p(x,s)\,\mathrm{d}x\,, \qquad (22)$$

where L is the length of the beam. Considering the loading types defined in Eqs. (18) and (20), the loading term $N_r(x,s)$, after some algebra, can be shown to read

$$N_r(x,s) = F(s)\varphi_r(x_F) - C_m C_q W_r s Z_{el}(s)\eta_r(s)\,, (23)$$

where

$$W_r = \left[\frac{\partial\varphi_r(b)}{\partial x} - \frac{\partial\varphi_r(a)}{\partial x}\right]\sum_{s=1}^{n}\left[\frac{\partial\varphi_s(b)}{\partial x} - \frac{\partial\varphi_s(a)}{\partial x}\right]. (24)$$

6. FREQUENCY RESPONSE MODEL

Considering the equation of motion defined in Eq. (21) and since the second loading term in the right hand side of Eq. (23), which represents the effects of the direct piezoelectric effect and the passive shunted electrical network on the net response of the beam, only depends of the modal coordinate $\eta(s)$, Eq. (21) can be rewritten as

$$\left[s^2 + 2\xi\omega_r s + \omega_r{}^2 + Z_r(s)\right]\eta_r(s) = F(s)\varphi_r(x_F)\,, (25)$$

where

$$Z_r(s) = C_m C_q W_r s Z_{el}(s)\,. \qquad (26)$$

The correspondent frequency response function (FRF) can be readily obtained from Eq. (25) by dividing $\eta_r(s)$ by $F(s)$ and considering Eq. (2) which, also defining $s = \mathrm{j}\omega$, yields

$$\frac{W(\mathrm{j}\omega)}{F} = \sum_{r=1}^{\infty}\frac{\varphi_r(x_F)\varphi_r(x)}{(\omega_r{}^2 - \omega^2) + \mathrm{j}(2\xi_r\omega_r\omega) + Z_r(\mathrm{j}\omega)}\,, (27)$$

where $W(\mathrm{j}\omega)$ and F are the complex displacement and disturbance force amplitudes.

7. CASE STUDY

To demonstrate the damping performance of shunting piezoelectric transducers, a case study comprising a clamped aluminum beam, with dimensions $15\times2\times400$ mm, and piezoelectric patch (PZT), with dimensions $15\times1\times30$ mm, mounted 5 mm away from the clamped edge, was considered. The material properties of the electromechanical system are presented in Tab. 1.

Table 1: Material properties of the beam and PZT.

Aluminum		PZT	
E_b[Pa]	70×10^9	e_{31}[C m^{-2}]	-3.16
ρ_b[Kg m^{-3}]	2710	ϵ_{33}^T [F m^{-1}]	0.98×10^{-8}

The direct FRF (receptance) at the beam free tip is presented in Fig. 2 with and without the passive control, when a resistance tuned to damp the first mode is used as shunt impedance. As can be seen, the damping strategy has damped the first mode significantly, as expected, and a small frequency shift, increasing the resonance frequency of both modes, has been produced.

Some care must be taken though in the generalization of these conclusions to a real implementation of the proposed damping strategy. Real physical constraints regarding the value of the resistance and the non-perfect (ideal) behavior of the materials and electronic network may pose some difficulties and the damping efficiency may be compromised.

Other alternative shunt strategies, using more complex single- and multi-mode damping strategies employing real or synthesized inductances and negative capacitances, can be seen as more efficient means of dissipating energy. The present model through the generic impedance term $Z_{sh}(s)$ can still model any arbitrary shunted impedance and therefore can successfully be used to capture the underlying phenomenological effects.

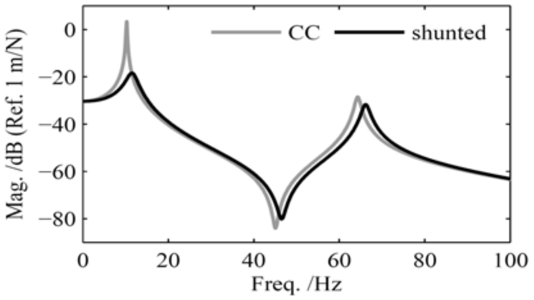

Fig 2: Free tip direct receptance of the beam in closed-circuit (CC) and with resistive shunt control.

8. CONCLUSION

The proposed phenomenological model has been shown to capture the damping effects of shunted piezoelectric transducers and its applicability was successfully shown to yield a good ideally damping performance.

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